Chapter 15

More Graphs—Eulerian, Bipartite, and Colored

15.1 Eulerian graphs

Ever seen those puzzles that ask you to trace some shape without lifting the pencil off the paper? For graph theory initiates such questions present no difficulty, separating this select elite from the rest of the human race who are doomed to spend their Sunday afternoons hunched over, putting page after page out of commission, searching in vain for the ever-elusive drawing.

Given a graph $G = (V, E)$, define a *tour* of G as a walk $T = (v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_{n+1})$ in G, such that T does not trace any edge more than once. (That is, $e_i \neq e_j$ for all $1 \leq i < j \leq n$.) The tour is said to be *Eulerian* if, in addition, $v_{n+1} = v_1$, $V(T) = V$, and $E(T) = E$. Thus an Eulerian tour traverses all the edges of G, "walking along" each exactly once, eventually coming back to where it started. (Particular vertices may and generally will be visited more than once.) A graph is said to be Eulerian if and only if it has an Eulerian tour.

Eulerian graphs were discussed by the great Leonhard Euler, the most prolific mathematician of all time. Euler's analysis of these graphs, presented in 1736, marks the birth of graph theory.

Theorem 15.1.1. *A graph is Eulerian if and only if it is connected and each of its vertices has even degree.*

Proof. We first prove that if G is Eulerian its vertices all have even degree. Indeed, trace an Eulerian tour of G starting and ending at a vertex v. Every time the tour enters an intermediate vertex it also leaves it along a different edge. In the very first step the tour leaves v and in the last step it enters v. Thus we can label the edges incident to any vertex as "entering" and "leaving", such that there is a bijection between these two sets. This shows that the degree of every vertex is even.

To prove that a graph $G = (V, E)$ with all vertex degrees being even is Eulerian, consider the longest tour $T = (v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_{n+1})$ in G. (The length of a tour is measured by its number of edges.) We prove below that T is Eulerian. Namely, we prove that:

(a) $v_1 = v_{n+1}$

(b) $n = |E|$

Proof of (a). Assume for the sake of contradiction that $v_1 \neq v_{n+1}$. Then the number of edges of T incident to v_1 is odd. (After T first leaves v_1, it enters and leaves it an even number of times.) Since the degree of v_1 in G is even, there is an edge e of G that is incident to v_1 but not part of T. We can extend T by this edge, obtaining a contradiction.

Proof of (b). We can assume that $v_1 = v_{n+1}$. Suppose $V(T) \neq V$. Consider a vertex $v \in V \setminus V(T)$ and a vertex $u \in V(T)$. Since G is connected, there is a path P between v and u in G. Consider the first time a vertex of T is encountered along P; this vertex is v_i for some $1 \leq i \leq n$. Let $e' = \{v', v_i\}$ be the edge along which P arrives at v_i and note that $v' \notin V(T)$. This implies that we can augment T by v' and e', and obtain a longer tour T', namely

$$T' = (v', e', v_i, e_i, \ldots, v_n, e_n, v_1, e_1, \ldots, v_{i-1}, e_{i-1}, v_i).$$

We have reached a contradiction and can therefore assume that $V(T) = V$. That is, T visits all the vertices of G. Assume for the sake of contradiction that $E(T) \neq E$, so there exists an edge $e' = \{v_i, v_j\}$ of G, for some $1 \leq i < j \leq n$, that is not part of T. Then we can augment T by the edge e', and obtain a longer tour T', namely

$$T' = (v_i, e', v_j, e_j, v_{j+1}, e_{j+1}, \ldots, v_n, e_n, v_1, e_1, \ldots, v_i, e_i, \ldots, v_{j-1}, e_{j-1}, v_j).$$

T' is longer than T by one edge, which is a contradiction that proves the theorem.

$\square$

Proof technique: Considering an extremal configuration. In the above proof the crucial idea was to consider the longest tour in the graph. This is an instance of a common proof technique: If we need to prove that some configuration with particular properties exists (like an Eulerian tour), consider the *extremal* (longest, shortest, etc.) configuration of a related type (usually one that has some but not all of the required properties), and prove that this extremal configuration has to satisfy *all* of the required properties. Some steps in the proof usually proceed by contradiction: If the extremal configuration wasn't of the required type we could find a "more extremal" one, which is a contradiction.

15.2 Graph coloring

Consider a wireless company that needs to allocate a transmitter wavelength to each of its users. Two users who are sufficiently close need to be assigned different wavelengths to prevent interference. How many different wavelengths do we need? Of course, we can just assign a new wavelength to every user, but that would be wasteful if some users are far apart. So what's the least number of wavelengths we can get away with?

We can model the users as vertices in a graph and connect two vertices by an edge if the corresponding users are sufficiently close. A *coloring* of this graph $G = (V, E)$ is an assignment of colors to vertices, such that no two adjacent vertices get the same color. The above question can now be restated as asking for the minimum number of colors that are needed for a coloring of G.

Let us be a bit more precise in defining colorings: A *k-coloring* of G is said to be a function $c : V \to \{1, 2, \dots, k\}$, such that if $\{v, u\} \in E$ then $c(v) \neq c(u)$. The smallest $k \in \mathbb{N}$ for which a k-coloring of G exists is called the *chromatic number* of G. If a k-coloring of G exists, the graph is said to be k-colorable. There are many deep results concerning colorings and the chromatic number. At this point we only give the simplest one:

Proposition 15.2.1. *If the degree of every vertex in a graph G is at most k, then the chromatic number of G is at most $k + 1$.*

Proof. By induction on the number of vertices in G. (The degree bound k is fixed throughout the proof.) If G has a single vertex, then the maximal degree is 0 and the graph is 1-colorable. Since $1 \leq k+1$, the proposition holds. Suppose every graph with at most n vertices and all vertex degrees at most k is $(k+1)$-colorable. Consider a particular graph $G = (V, E)$ with $n + 1$ vertices, and all degrees at most k. Let G' be the graph obtained from G by deleting a particular vertex v and all the edges incident to v. That is, G' is the incident subgraph of G on the vertices $V \setminus \{v\}$. G' has n vertices, all of degree at most k, and is thus $(k + 1)$-colorable. Let c' be such a coloring of G'. We extend it to a coloring c of G as follows. For every vertex $u \in G$ such that $u \neq v$ we define $c(u) = c'(u)$. The vertex v has at most k neighbors in G and there is at least one color i among $\{1, 2, \dots, k + 1\}$ that has not been assigned to any of them. We define $c(u) = i$. This is a $(k + 1)$-coloring, and the proposition follows. □

15.3 Bipartite graphs and matchings

A bipartite graph is a graph that can be partitioned into two parts, such that edges of the graph only go between the parts, but not inside them. Formally, a graph $G = (V, E)$ is said to be bipartite if and only if there exist $U \subseteq V$, such that

$$E \subseteq \{\{u, u'\} : u \in U \text{ and } u' \in V \setminus U\}.$$

The sets U and $V \setminus U$ are called the *classes* of G. A *complete bipartite graph* $K_{m,n}$ is a graph in which all the edges between the two classes are present. Namely, $K_{m,n} = (V, E)$, where $V = \{1, 2, \dots, m+n\}$ and $E = \{\{i, j\} : 1 \leq i \leq m, m+1 \leq j \leq m+n\}$. The number of edges in K_n is mn. From the definition of coloring, it follows that a graph is bipartite if and only if it is 2-colorable. (Check!) Here is another useful characterization of bipartite graphs:

Proposition 15.3.1. *A graph is bipartite if and only if it contains no cycle of odd length.*

Proof. For one direction of the claim, let G be a bipartite graph and let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a cycle in G. Suppose without loss of generality that $v_1 \in U$, where U is as in the definition of bipartiteness. Then by simple induction that we omit, $v_i \in U$ for every odd $1 \leq i \leq n$. Since $\{v_n, v_1\} \in E$, $v_n \in V \setminus U$ and thus n is even. It follows that the number of edges in C is even.

Before proving the other direction, we need a simple lemma.

Lemma 15.3.2. *Given a graph $G = (V, E)$, let $P = (v_1, v_2, \ldots, v_n)$ be a shortest path between two vertices v_1 and v_n in G. Then for all $1 \leq i < j \leq n$, $P_i = (v_i, v_{i+1}, \ldots, v_j)$ is a shortest path between v_i and v_j.*

Proof. Proof by contradiction. Let $Q_i = (v_i, u_1, u_2, \ldots, u_l, v_j)$ be a shortest path between v_i and v_j. Assume for the sake of contradiction that Q_i is shorter than P_i. Consider the walk

$$Q = (v_1, v_2, \ldots, v_i, u_1, u_2, \ldots, u_l, v_j, v_{j+1}, \ldots, v_n).$$

Since Q_i is shorter than P_i, Q is shorter than P. Now consider the graph $G' = (V(Q), E(Q))$. This graph is connected, and thus there is a shortest path P' between v_1 and v_n in G'. The number of edges in this shortest path cannot exceed the total number of edges in G', and thus P' is shorter than P. Since P' is also a path between v_1 and v_n in G, we have reached a contradiction. $\square$

We now turn to the other direction of the proposition. Assume that $G = (V, E)$ has no odd cycle. If G has more than one connected component we look at every component separately. Clearly, if every component is bipartite, G as a whole is bipartite. Thus assume that G is connected. Pick an arbitrary vertex $v \in V$ and define a set $U \subseteq V$ as

$$U = \{x \in V : \text{ the shortest paths from } v \text{ to } x \text{ have even length}\}.$$

Clearly, $V \setminus U$ is the set

$$V \setminus U = \{x \in V : \text{ the shortest paths from } v \text{ to } x \text{ have odd length}\}.$$

We prove that no two vertices in U are adjacent; the proof for $V \setminus U$ is similar. Consider for the sake of contradiction an edge $e = \{u, u'\} \in E$, such that $u, u' \in U$. Denote a shortest path from v to u by P_1 and a shortest path from v to u' by P_2. Given two vertices s and t on a path P, let $P^{s,t}$ be the part of P that connects s and t. Consider a vertex w that lies on both P_1 and P_2. The above lemma implies that $P_1^{v,w}$ and $P_2^{v,w}$ are shortest paths between v and w and thus have the same length, which we denote by $l(w)$. Consider the vertex w^* shared by P_1 and P_2 that maximizes $l(w)$ among all such w. The paths $P_1^{w^*,u}$ and $P_2^{w^*,u'}$ share no vertex in common other than w^*. Furthermore, the length of $P_1^{w^*,u}$ is the length of P_1 minus $l(w^*)$ and the length of $P_2^{w^*,u'}$ is the length of P_2 minus $l(w^*)$. Since the lengths of P_1 and P_2 are both even, the lengths of $P_1^{w^*,u}$ and $P_2^{w^*,u'}$ have the same parity (that is, they are either both even or both odd). Now consider the cycle C composed of $P_1^{w^*,u}$, the edge $\{u, u'\}$, and $P_2^{w^*,u'}$. Since $P_1^{w^*,u}$ and $P_2^{w^*,u'}$ share no vertex in common other than w^*, C really is a cycle in G. Moreover, since the lengths of $P_1^{w^*,u}$ and $P_2^{w^*,u'}$ have the same parity, the number of edges in C is odd! We have reached a contradiction that completes the proof. $\square$

Bipartite graphs are particularly useful to model symmetric relations from one set to another. For example, given a collection of boys and girls, we could model the relation "wants to go to the prom with" by a bipartite graph. Given such preferences, an interesting question is whether we can pair the boys up with the girls, so that they all end up going to the prom with someone they actually want to go with. It turns out that this question has a very precise answer. To state the theorem we need to define the notion of matching:

Definition 15.3.3. *Given a bipartite graph $G = (V, E)$, a* matching *B in G is a set of disjoint edges. Namely, $B \subseteq E$ and $e_1 \cap e_2 = \emptyset$ for any $e_1, e_2 \in B$. A matching is said to be* perfect *if $\bigcup_{e \in B} e = V$.*

Consider now a set B of boys, a set G of girls, and a symmetric relation P from B to G. Define a graph $W = (B \cup G,\ \{\{b, g\} : (b, g) \in P\})$. The above question simply asks to characterize when there exists a perfect matching in W. The below result, known as Hall's theorem, provides such a characterization. To state the theorem, we use another piece of notation: Given a subset S of the vertices of W, we let $\Gamma(S)$ be the set of vertices of W adjacent to at least one of the vertices of S.

Theorem 15.3.4. *A bipartite graph $W = (V, E)$ with classes B and G has a perfect matching if and only if $|B| = |G|$ and $|\Gamma(S)| \geq |S|$ for all $S \subseteq B$.*

Proof. One direction is easy: Assume W has a perfect matching and consider a set $S \subseteq B$. Every element of S is matched to a distinct element of G and hence $|\Gamma(S)| \geq |S|$. In particular, $|G| \geq |B|$. By a symmetric argument we get that $|B| \geq |G|$ and thus $|B| = |G|$.

For the other direction, assume that $|B| = |G|$ and that $|\Gamma(S)| \geq |S|$ for all $S \subseteq B$. We prove that there exists a perfect matching in W by strong induction on $|B|$. For the base case, if $|B| = |G| = 1$, the matching consists of the single edge of W. Assuming that the claim holds for all graphs with $|B| \leq k$, consider a graph W as above with $|B| = k + 1$. We distinguish between two possibilities:

(a) If for every $S \subset B$, $|\Gamma(S)| > |S|$, we take an arbitrary $x \in B$ and match it with an adjacent $y \in G$. Then for every subset S' of $B \setminus \{x\}$, it still holds that the number of vertices of $G \setminus \{y\}$ adjacent to at least one of the vertices of S' is at least $|S'|$. We can thus match the vertices of $B \setminus \{x\}$ with the vertices of $G \setminus \{y\}$ by the induction hypothesis.

(b) If for some $S \subset B$, $|\Gamma(S)| = |S|$, we note that for every $S' \subseteq S$, the number of vertices in $\Gamma(S)$ adjacent to at least one of the vertices of S' is at least $|S'|$.

Thus we can match S with $\Gamma(S)$ by the induction hypothesis. Now we need to show that we can match $B \setminus S$ with $G \setminus \Gamma(S)$. Consider a set $S' \subseteq B \setminus S$ and the set T' of its neighbors in $G \setminus \Gamma(S)$. Note that the set of neighbors of $S \cup S'$ in G is $\Gamma(S) \cup T'$. Thus $|S \cup S'| \leq |\Gamma(S) \cup T'|$. Since $|S| = |\Gamma(S)|$, we get that $|S'| \leq |T'|$. Thus by the induction hypothesis we can also match $B \setminus S$ with $G \setminus \Gamma(S)$.

This shows that in both cases all the vertices of B can be matched with vertices of G as required, and concludes the proof. $\square$